

ON THE GLOBAL DIMENSION OF
CATEGORIES OF DIAGRAMS

By

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INTRODUCTION

Let I be a finite partially ordered set of n elements, and let R be a ring. Then we say form the "pattern ring" $R(I)$, which is defined to be that subring of the ring of $I \times I$ matrices consisting of those matrices (x_{ij}) for which $x_{ij} = 0$ whenever $i \not\leq j$.

Originally, we set out to give necessary and sufficient conditions on I in order that $\text{gl.dim } R(I) \leq n$ in the special case where R is a field. This problem was posed to us by Professor Clark, who had solved the problem for $n = 1$ (Pacific J. math. (1) 83(1967), 453-471). Some time after beginning work on this problem, Professor Clark received a letter from Harry Mitchell informing him of his more general results. In his 1969 dissertation, Mitchell had considered the following problem. Let R be an abelian category with enough projectives, and let $[I, R]$ denote the category of diagrams in R over I . In this case, Mitchell asked for necessary and sufficient conditions on I such that $\text{gl.dim } [I, R] \leq n + \text{gl.dim } R$.

Mitchell had in fact solved this problem for $n = 1$ in his dissertation. He further showed that when R is the category R_A of right R -modules, then $[I, R_A]$ is equivalent to the category $R_{A(I)}$ of right $R(I)$ -modules, hence

ql, for $[1, \mathcal{A}_k] = \text{ql dim } \mathcal{A}(1)$.

In his dissertation [see also [1]], Mitchell raised the question whether or not the integer $\text{ql dim } [1, \mathcal{A}] = \text{ql dim } \mathcal{A}$ depends only on $\mathcal{A}$. If so, he called this unique number the dimension of $\mathcal{A}$. Mitchell characterized those $\mathcal{A}$ of dimension 1 (and 0) for abelian categories with enough projectives. In fact, more recently, through the use of the Torricelli definition of ql dim , he has extended his results to the case in which $\mathcal{A}$ is any abelian category [1].

Moreover, in his letter to Professor Clark, Mitchell essentially stated a characterization of those $\mathcal{A}$'s of dimension 2. Actually, he left out a small, but important, detail which he has recently supplied in a letter to the author. However, the statement was given without proof, and the author's attempt to provide one led to the results of Chapter IV and to a slightly different characterization.

The current statement of Mitchell's result is provided in Chapter IV for comparison with the author's result. The author would like to point out that he was greatly aided by his knowledge of Mitchell's characterizations, which was only slightly incomplete.

In any event, those partially ordered sets of dimension 2 have been characterized. The complexity of these characterizations, as compared to those for dimensions 0 and 1, would seem to indicate that a characterization for dimen-

sion β would be extraordinarily complex, or impossible. In fact, we discovered that such a characterization is impossible while trying to prove that a certain complex type of partially ordered set was of dimension 4. This class of partially ordered sets actually splits into two subclasses, one of which seems to be a class of noncharacterizable and the other of which seems to be a class of partially ordered sets of dimension 4. A partially ordered set β is given such that $\text{gl.dim } [1, \beta_0] = 1 + \text{gl.dim } [1, \beta]$, where β_0 and β are categories of matrix spaces over fields of characteristic equal to 2 and not equal to 2, respectively.

In Chapters II and III, we develop some machinery for finding $\text{gl.dim } [1, \beta]$. This machinery is used in Chapter IV, although more direct proofs for the results of Chapter IV exist. In Chapter II, we show that $\text{gl.dim } [1, \beta]$ may be found by calculating the projective dimension of diagrams which have a minimum object of at most one vertex. (These interested in ring theory will perhaps note the similarity to the fact that the global dimension of an Artinian ring is the supremum of the homological dimensions of the simple modules.) Also in Chapter II, we give a method for exploiting this result. We show that one may use an "n-dimensional" resolution, as applied to the usual projective resolution, to find the homological dimension for these simple diagrams. The idea of the "n-dimensional" resolution may be found extensively in [1].

To further utilize this method, we introduce in Chapter III the idea of basis diagrams and show how the homological dimension of basis diagrams may be easily computed (non-partially!). Also, we point out what we believe is the main of basis diagrams as the "n-dimensional" realizations which we give in Chapter II.

In Chapter IV, we characterize those partially ordered sets of dimension 2. The reader will notice that one encounters the most difficulty in establishing a sequence of propositions on finite partially ordered sets.

In Chapter V, we exhibit the counterexample mentioned above, which we believe to be the smallest such.

CHAPTER I

PRELIMINARIES

We shall use the notation of [2] throughout, unless otherwise stated. Also, we shall assume that the reader has a basic understanding of Chapters I and II of the same. In this chapter, we collect from [1] and [2] some of the definitions and results which are most important for our purposes.

(1.1) Let $(I, \leq)$ be a finite partially ordered set. Then $(I, \leq)$ may be considered as a category having a single morphism from i to j if and only if $i \leq j$. If $\mathcal{B}$ is a category, then by a diagram in $\mathcal{B}$ over I we mean a functor $B: I \rightarrow \mathcal{B}$. The category $(I, \mathcal{B})$ is defined to be the class of all diagrams in $\mathcal{B}$ over I . For B and B' in $(I, \mathcal{B})$, take $[B, B']$ to be the set of all natural transformations from B to B' . Then an object B in $(I, \mathcal{B})$ is a family of objects $\{B_i\}$, $i \in I$, in $\mathcal{B}$ together with morphisms $B_{ij}: B_i \rightarrow B_j$ defined whenever $i \leq j$, and satisfying $B_{jk}B_{ij} = B_{ik}$ (for $i \leq j \leq k$) and $B_{ii} = 1_{B_i}$. A morphism $f: B \rightarrow B'$ is a family $f_i: B_i \rightarrow B'_i$ satisfying $B'_{ij}f_j = f_iB_{ij}$ whenever $i \leq j$.

(1.2) Throughout this dissertation, $\mathcal{B}$ will consistently denote a fixed, but arbitrary, abelian category with

enough projectives. The letter $\mathbf{I}$ will always denote a $\mathbf{B}$ -cube not partially ordered by $\leq$. After some justification (see 1.7), we will always assume that $\mathbf{I}$ has a greatest and a least element.

(1.3) We recall some basic properties of $[I, \mathbf{B}]$ as found in [2, pp. 237-241]. $[I, \mathbf{B}]$ is an abelian category and if $\mathbf{B}$ has enough projectives, then so does $[I, \mathbf{B}]$. Moreover, $f: \mathbf{B} \rightarrow \mathbf{B}'$ is an epimorphism (monomorphism) if and only if f_i is an epimorphism (monomorphism) for each i . The morphism $g: \mathbf{B} \rightarrow \mathbf{B}'$ is a kernel of f if and only if $g_i: \mathbf{B}_i \rightarrow \mathbf{B}'_i$ is a kernel of $f_i: \mathbf{B}_i \rightarrow \mathbf{B}'_i$ for all i . Also, the morphism $g: \mathbf{B} \rightarrow \mathbf{B}'$ is a cokernel of f if and only if $g_i: \mathbf{B}_i \rightarrow \mathbf{B}'_i$ is a cokernel of f_i for all i . If $\mathbf{B} = \bigoplus_{i \in I} \mathbf{B}^i$, then $\pi_i = \bigoplus_{j \in I} \pi_{ij}^i$.

(1.4) Mitchell [2] characterized projectives in $[I, \mathbf{B}]$ as follows: define a functor $\mathbf{B}_i: \mathbf{B} \rightarrow [I, \mathbf{B}]$ by letting, for k in $\mathbf{B}$, $\mathbf{B}_i(k)_j = k$ for $j = i$, and $\mathbf{B}_i(k)_j = 0$ otherwise. The morphisms $\mathbf{B}_i(\mathbf{B})_{jB}$ are defined in the obvious way, as in $\mathbf{B}_i(f)$ for $f: \mathbf{B} \rightarrow \mathbf{B}'$. Mitchell showed that $\mathbf{B}$ is projective in $[I, \mathbf{B}]$ if and only if $\mathbf{B} = \bigoplus_{i \in I} \mathbf{B}_i(\mathbf{P}_i)$, where $\mathbf{P}_i$ is projective for all i . If there is more than one partially ordered set involved, then the notation $\mathbf{B}_i^{\mathbf{B}}$ is used to avoid confusion.

Other than the above, $\mathbf{B}_i$ has a very important property. For $u: \mathbf{B} \rightarrow \mathbf{B}'$ where $\mathbf{B} \in \mathbf{B}$ and $\mathbf{B}' \in [I, \mathbf{B}]$, there is a

unique $\alpha \in \mathbb{Z}_2[X] \rightarrow \beta$ such that $\ell_1 = \alpha \ell_2$.

(1.5) We shall use the following notations

(1) For j in $\mathbb{Z}$, $[j] = \{i \in \mathbb{Z} : i \equiv j\}$.

(2) For $a \neq x \in \mathbb{Z}$, $[x] = \bigcup_{y \neq x} [y]$.

(3) For j in $\mathbb{Z}$, $u(j) = \{i \in \mathbb{Z} : i \equiv j\}$.

(4) For $a \neq x \in \mathbb{Z}$, $u(x) = \bigcup_{y \neq x} u(y)$.

(5) For $i \equiv j$ in $\mathbb{Z}$, $\langle i, j \rangle = \{k \in \mathbb{Z} : i \equiv k \equiv j\}$.

(6) For $x \in \mathbb{Z}$, $\text{max } x = \{j \in \mathbb{Z} : k \geq j \Rightarrow k \notin x\}$.

(7) For $x \in \mathbb{Z}$, $\text{min } x = \{j \in \mathbb{Z} : k \leq j \Rightarrow k \notin x\}$.

(8) For $x \in \mathbb{Z}$, $[x] = \text{max } x$.

(1.6) The results of Mitchell [1] and [2] in this section are particularly relevant in what follows.

(1.6.1) An ordered subset T' of T is said to be a normal subset if

$$|\max \{ \langle x, y \rangle \cap T' \}| = 1$$

for all i in $u(T')$ and j in T' ($i \neq j$).

(1.6.2) Theorem. [Mitchell] If T' is a normal subset of T , then $\text{gl.dim } [T', \mathbb{Z}] = \text{gl.dim } [1, \mathbb{Z}]$.

(1.6.3) An ordered subset T' of T is said to be a restricted subset if

$$|\max \{ \langle x, y \rangle \cap T' \}| = 1$$

for all i in I' and j in J ($i \neq j$).

(1.6.4) Theorem. (Mitchell) If I' is a separated subset of I , then $q_1 \text{dim } (I', B) = q_1 \text{dim } (I, B)$.

(1.6.5) Theorem. (Mitchell) Let $J = \min I$. Then

$$q_1 \text{dim } (I, B) = \sup_{j \in J} q_1 \text{dim } (i(j), B).$$

(1.6.6) Theorem. (Mitchell) Let $J = \max I$. Then

$$q_1 \text{dim } (I, B) = \sup_{j \in J} q_1 \text{dim } (i(j), B).$$

(1.6.7) Lemma is the proof of 1.6.5. Let

$J = \min I$ and $r_j: (I, B) \rightarrow (i(j), B)$ be the restriction function. Then for i in I, B , let $D = \sup_{j \in J} \text{hd } r_j(i)$. [See (8, p. 232).]

(1.7) Proof. Let $J = \min I$ and $J' = \max I$. Then by 1.6.5 and 1.6.6 we have

$$q_1 \text{dim } (I, B) = \sup_{j \in J} q_1 \text{dim } (i(j), B).$$

Hence from this point on, it will be assumed that I has a least and a largest element.

CHAPTER II

A FIRST APPROXIMATION TO THE GLOBAL CONDITION OF (1.8)

(2.1) In this section, we give a theorem which enables us to find $\text{gl.dim } [1, R]$ by calculating $\text{hd } R$ for disjuncts R of a relatively simple structure.

(2.1.1) Definition. Define the function $R_1: R \rightarrow [1, R]$ for R in I as follows. For R in I , let $R_1(R)_1 = 0$ for $1 \notin R$, and $R_1(R)_1 = R$. The morphisms $R_1(R)_{2R}$ are necessarily 0. Let $R_1(f)$, where $f: R \rightarrow R'$, be defined in the obvious way.

(2.1.2) Theorem. We have

$$\text{gl.dim } [1, R] = \sup \{ \text{hd } R_1(R) : 1 \in R, R \in R \}.$$

Proof. Clearly,

$$\text{gl.dim } [1, R] \geq \sup \{ \text{hd } R_1(R) : 1 \in R, R \in R \}.$$

We will establish the reverse inequality by showing that the following is true. For R in $[1, R]$, we have

$$(1) \quad \text{hd } R \leq \sup_1 \text{hd } R_1(R_1).$$

Let $n(R) = \{i \in I \mid D_i \neq 0\}$. We prove (1) by induction on $|n(R)|$. Clearly if $|n(R)| = 1$, then we are done. So assume that (1) holds for $R' \in [T, R]$ such that $|n(R')| < |n(R)|$. Let $k \in \min n(R)$. Then define $d: R \rightarrow R_k \oplus R_k$ by $d_i = 1_{D_k}$, and $d_i = 0$ for $i \neq k$. It is easily verified that d is well defined; and in fact, d is an isomorphism. Let $\eta: R' \rightarrow 0$ be a kernel for d . It follows that $D_k^1 = D_k$ for $i \neq k$, and $D_k^1 = 0$. Thus $|n(R')| = |n(R)|$. By the induction hypothesis we have

$$(2) \quad \text{hd } R' \leq \sup_i \text{hd } R_i[D_k^1] = \sup_{i \neq k} \text{hd } R_i[D_k].$$

But we have the exact sequence

$$0 \longrightarrow R' \longrightarrow R \longrightarrow R_k \oplus R_k \longrightarrow 0,$$

from which we have

$$(3) \quad \text{hd } R \leq \sup \{ \text{hd } R', \text{hd } R_k \oplus R_k \}$$

It is easy to see that (2) and (3) give us (1).

(2.2) There are some results we need in the next section. Let the sequence

$$(2.2.1) \quad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be exact. Then for $n \in \mathbb{N}$, we have the exact sequence

$$\text{Ext}^{2n-1}(B, X) \longrightarrow \text{Ext}^{2n-1}(A, X) \longrightarrow \text{Ext}^{2n-2}(C, X)$$

and

$$\text{Ext}^{2n-1}(A, X) \longrightarrow \text{Ext}^{2n-2}(C, X) \longrightarrow \text{Ext}^{2n-2}(B, X),$$

where $n \in \mathbb{N}$. From these we get

$$(2.2.2) \quad \text{hd } A = \sup \{ \text{hd } B, \text{hd } C + 1 \},$$

and

$$(2.2.3) \quad \text{hd } C + 1 = \sup \{ \text{hd } A, \text{hd } B + 1 \}.$$

(2.2.4) Proposition. Given 2.2.1, if $\text{hd } B = \text{hd } C$ or $\text{hd } B < \text{hd } A$, then $\text{hd } A = \text{hd } C + 1$.

Proof. If $\text{hd } B < \text{hd } A$, then $\text{hd } A = \text{hd } C + 1$ by [2, Prop. 5.1, p. 100]. On the other hand, if $\text{hd } B = \text{hd } C$, then by 2.2.2 $\text{hd } A < \text{hd } C + 1$, and by 2.2.3 $\text{hd } C + 1 < \text{hd } A$.

(2.2) We now give a method for determining $\text{hd } H_p(A)$ without using a projective resolution. Define K^0 to be $H_0(A)$, and K^1 to be $\bigoplus (H_j(A) - j \neq 0) \oplus \{e(1), 2\}$. Then letting K^0 and K^1 be the obvious kernels, we have the exact sequence

$$(1) \quad 0 \longrightarrow K^0 \longrightarrow K^1 \xrightarrow{d^1} H_0(A) \longrightarrow 0$$

and

$$(2) \quad 0 \longrightarrow K^1 \longrightarrow K^2 \xrightarrow{d^2} K^0 \longrightarrow 0.$$

Now for $p-1 \neq 1$, assume that we have the exact sequence

$$(3) \quad 0 \longrightarrow K^{p-1} \longrightarrow K^{p-2} \xrightarrow{d^{p-2}} K^{p-3} \longrightarrow 0$$

($K^{p-3} \neq 0$). We define K^p to be $\bigoplus_{j \neq p} H_j(K^{p-1})$, and K^p to be the kernel of the morphism d^p induced by the morphism $l_{K^{p-1}}$. Thus we have the exact sequence

$$(p+1) \quad 0 \longrightarrow K^p \longrightarrow K^p \xrightarrow{d^p} K^{p-1} \longrightarrow 0.$$

(2.2.2) Moreover, we have $\text{hd } K^p = \text{hd } A$ for all p such that $K^p \neq 0$.

Lemma. Since $\text{hd } K_j(R) = \text{hd } R$ for all j in $\mathbb{N}$ (see [1, § 3]), the result is clear for $p = 0$ and for $p = 1$. To complete the proof, we show that $\text{hd } K_1^p = \text{hd } R$ when $K_1^p \neq 0$. Because this result is held for $p \geq 1$, i.e., $p-1 \geq 0$, we consider the sequence

$$\bigoplus_{k \in \mathbb{N}} K_k^{p-1} \xrightarrow{\alpha} \left(\bigoplus_{k \in \mathbb{N}} K_k^{p-1} \right) \oplus K_0^{p-1} \xrightarrow{\beta} K_0^{p-1},$$

where $\alpha = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}$, γ is the identity on $\bigoplus_{k \in \mathbb{N}} K_k^{p-1}$, and β is the restriction of β_0^p to $\bigoplus_{k \in \mathbb{N}} K_k^{p-1}$. By a straightforward argument, one may show that α is a kernel for β_0^p .

Since α is a kernel for β_0^p , we must have $K_0^p \subseteq \bigoplus_{k \in \mathbb{N}} K_k^{p-1}$. But $\text{hd } \bigoplus_{k \in \mathbb{N}} K_k^{p-1} = \sup_k \text{hd } K_k^{p-1} = \text{hd } R$ if some $K_k^{p-1} \neq 0$ (see Proposition 2.3.1).

(2.3.2) **Proposition.** Suppose $\text{hd } K_1(R) > \text{hd } R$ (which by [2, Lemma 50.4, p. 275] is true if $\pi \neq \pi^2$). Then if p is the smallest integer such that $\text{hd } R^p = \text{hd } R$, we have $\text{hd } K_1(A) = \text{hd } R + p + 1$.

Proof. First observe that if $p > q \geq 0$, then $\text{hd } R^q > \text{hd } R$, which is shown as follows. Let q be the smallest integer such that $\text{hd } R^q < \text{hd } R$. Now q is not 0, since by 2.2(4)

$$\text{hd } R^0 = \text{hd } K_1(A) - 1 > \text{hd } R,$$

thus we have by the minimality of q that $\text{hd } R^{q-1} > \text{hd } R$ and since $p > q-1$, we have $\text{hd } R^{p-1} > \text{hd } R$. Hence it follows from 2.2(4) that

$$\text{hd } R^q = \text{hd } R^{p-1} - 1 > \text{hd } R,$$

which is a contradiction. By repeatedly applying 2.3.4 we have

$$\text{hd } R_1(k) = \text{hd } R^{p-1} = (\text{hd } R^{1+1}) + 1 = \dots = \text{hd } R^{p+p+1}.$$

(2.4) It is apparent from 2.3.2 that a major point of interest in determining whether or not $\text{hd } R^p = \text{hd } R$, is, in this regard it is of particular importance that the kernels R^p are all networks of subobjects of R_m^p , where m denotes the greatest element of I (cf. 1.7).

(2.4.1) More precisely, we shall say that a diagram D in $[I, \mathcal{K}]$ is a network of subobjects (of some object k in $\mathcal{K}$) if each object D_i is a subobject of k such that whenever $i \neq j$, the inclusion morphisms σ_i and σ_j associated with D_i and D_j are related to R_{ij} by the commutative diagram

$$\begin{array}{ccc} D_i & \xrightarrow{\sigma_j} & k \\ \sigma_i \downarrow & & \uparrow \sigma_j \\ D_j & \xrightarrow{\sigma_i} & k \end{array}$$

In particular, each R_{ij} is a monomorphism. If I contains a greatest element m , then D is a network of subobjects (of R_m) if and only if each R_{ij} is a monomorphism.

We denote by $\mathcal{N}[I, \mathcal{K}]$ the full subcategory of $[I, \mathcal{K}]$ whose objects are the networks of subobjects.

(2.4.2) It now follows, m will always denote the greatest element of I (cf. 1.7). Let J be a subset of the complete elements of I such that $\{J\} \neq \emptyset$. We define a

partially ordered set $I^1 = I \cup \{a\}$, $a \notin I$, by letting the order remain the same for elements in I and defining for $x \in I$,

$$a \leq x \text{ iff } x \in \{x\},$$

and

$$a \leq x \text{ iff } \{x\} \subset \{a\}.$$

One easily checks that this does partially order I^1 . (Note, however, that if I has only one element, say $\{$, then $a \leq \{ \leq a$, so that antisymmetry fails to hold.)

(2.4.3) Consequently, if I is a set of incomparable elements of I , then we shall mean by the expression $I + \{x\}$ the partially ordered set defined in 2.4.2, if $|x| \neq \emptyset$, if $|x| = \emptyset$, then we shall understand $I + \{x\}$ to denote I .

(2.4.4) Define a functor $H: \mathcal{K}(I, \mathcal{K}) \rightarrow \mathcal{K}(I^1, \mathcal{K})$ as follows. For $\mathcal{D} \in \mathcal{K}(I, \mathcal{K})$, let $H(\mathcal{D})_{\mathcal{K}} = \mathcal{D}_{\mathcal{K}}$ for $\mathcal{K} \in I$; and let $H(\mathcal{D})_{\{a\}} = \bigcup_{\mathcal{K} \in I} \mathcal{D}_{\mathcal{K}}$ [the union is taken in $\mathcal{K}_{\mathcal{K}}$]. For the $\mathcal{D} \rightarrow \mathcal{D}'$, let $H(\mathcal{D})_{\mathcal{K}} = \mathcal{D}_{\mathcal{K}}$ for $\mathcal{K} \in I$; and let $H(\mathcal{D})_{\mathcal{K}}$ be the morphism induced by $f_{\mathcal{K}}$.

(2.4.5) It is straightforward to show that H preserves isomorphisms and monomorphisms.

$$(2.4.6) \text{ If } \mathcal{D} = \bigoplus_{1 \leq i \leq l} \mathcal{K}_i^2(\mathcal{K}_i), \text{ then } H(\mathcal{D}) = \bigoplus_{1 \leq i \leq l} \mathcal{K}_i^2(\mathcal{K}_i).$$

(2.4.7) Let us say that a sequence (not necessarily exact) $\mathcal{H}: \mathcal{K} \rightarrow \mathcal{K} \rightarrow \mathcal{K}$ is in $\mathcal{K}(I, \mathcal{K})$ if $\mathcal{K}, \mathcal{K}, \mathcal{K} \in \mathcal{K}(I, \mathcal{K})$.

and $P = \bigoplus_{i=1}^n E_i^T(A_i)$, by $H(X)$ we shall mean $H(X) \rightarrow H(P) \rightarrow H(X)$.

(2.4.2) If X is in $H(T, R)$, then $H(X)$ is in $H(T^1, R)$.

(2.4.3) It was observed in 2.3 that whether or not $\text{ad } X^P = \text{ad } X$ is of major interest in determining $\text{ad } H_n(A)$ [cf. 2.3]. Most of the material of the next chapter is intended to provide a practical means for settling this question. An essential result used in developing this material is that conditions may be placed on D [of H] so that if

$$H: D \rightarrow H \rightarrow P \rightarrow R \rightarrow Q$$

is exact, then so is

$$H(H), 0 \rightarrow H(X) \rightarrow H(P) \rightarrow H(R) \rightarrow 0.$$

CHAPTER III

CALCULATING THE DIMENSION OF BASIC COAGULATES

In this chapter, we always assume that $\mathcal{I}$ has a quasi-set and a least element $\{x\}$, $\{x\} \leq 1\}$.

(3.1) Remark: To the discussion of 2.3.2, we give the following definition.

(3.1.1) Definition. Let $\mathcal{J}$ be a set of incompatible elements of $\mathcal{I}$. We say the family $\{\alpha_j, j \in \mathcal{J}\}$ is quasi-independent if $\{x\} \leq 1$ and, assuming the definition has been made for $\mathcal{J}'$ such that $|\{\mathcal{J}'\}| < |\{\mathcal{J}\}|$, there is p in $\mathcal{J}$ such that

- (1) $\{\alpha_j, j \in \mathcal{J} \setminus p\}$ is quasi-independent,
- (2) $\{\alpha_j, j \in \mathcal{J}_p\}$ is quasi-independent, where $\mathcal{J}_p = \text{supp } \{(\mathcal{J} \setminus p) \cap \{x\}\}$, and
- (3) $\bigcup_{\mathcal{J} \setminus p} \alpha_j \cap \alpha_p = \bigcup_{\mathcal{J}_p} \alpha_j$.

We note that whenever $\mathcal{J}$ is a set of incompatible elements and $p \in \mathcal{J}$, we shall let $\mathcal{J}_p$ be as above throughout.

(3.1.2) Theorem: Let $\mathcal{J}$ be a set of incompatible elements of $\mathcal{I}$, let $\mathcal{I}^{\mathcal{J}} = \mathcal{I} \cup \{x\}$ (see 2.4.3), and let $\mathcal{H}: \mathcal{H}(\mathcal{I}, \mathcal{K}) \rightarrow \mathcal{H}(\mathcal{I}^{\mathcal{J}}, \mathcal{K})$ be the functor defined in 2.4.3. If

The exact sequence

$$0 \longrightarrow R \xrightarrow{f} P \xrightarrow{g} Q \longrightarrow 0$$

is in $\mathcal{B}(\mathbb{Z}, R)$, and $(Q_j : j \in \mathbb{Z})$ is quasi-independent. Then

$$R[R] : 0 \longrightarrow R[R] \longrightarrow P[R] \longrightarrow Q[R] \longrightarrow 0$$

is exact and is in $\mathcal{B}(\mathbb{Z}, R)$.

The proof of 3.1.2 is an easy consequence of the 3 Lemma, but we use the following lemma, since they are needed later. Of particular interest is

(3.1.3) THE 3-LEMMA. Consider

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R_1 & \xrightarrow{f_1} & R_2 & \xrightarrow{f_2} & R_3 & \longrightarrow & 0 \\
 & & \uparrow g_1 & & \uparrow g_2 & & \uparrow g_3 & & \\
 0 & \longrightarrow & P_1 & \xrightarrow{f_1} & P_2 & \xrightarrow{f_2} & P_3 & \longrightarrow & 0 \\
 & & \uparrow g_1 & & \uparrow g_2 & & \uparrow g_3 & & \\
 0 & \longrightarrow & R_1 & \xrightarrow{f_1} & R_2 & \xrightarrow{f_2} & R_3 & \longrightarrow & 0 \\
 & & \uparrow g_1 & & \uparrow g_2 & & \uparrow g_3 & & \\
 0 & \longrightarrow & P_1 & \xrightarrow{f_1} & P_2 & \xrightarrow{f_2} & P_3 & \longrightarrow & 0
 \end{array}$$

where all spaces are abelian. In addition, let the rows be exact and let each non-horizontal morphism be a monomorphism. Moreover, suppose that $P_1 \cap P_2 = P_1 \cap P_2 = P_1 \cup P_2$, and $P_2 = P_1 \cup P_2$. Then we have

$$R_2 / (R_1 \cup R_2) = (R_1 \cap R_2) / P_1.$$

Proof. To begin, let us note that we may consider $R_1, P_1, R_2, P_2, P_1, P_2, P_1$, and P_2 as subgroups of P_1 . Now let us show that $R_1 \cap R_2 = R_1$. Clearly $R_1 \subseteq R_1 \cap R_2$. But

$E_a \subset E_b$ and $E_b \subset E_c$. Therefore $E_a \cap E_b = E_a \cap E_c = E_a$. But $E_c(E_a \cap E_b) \subset E_c(E_a) = 0$. Since $E_a \cap E_b \neq \emptyset$ and $E_c = E_c$ we have then that $E_a \cap E_b = E_b$.

Next we note that by use of the dual of the 3 Lemma and the First Noether Isomorphism Theorem, we may reduce this lemma to the case $E_a = E_b = E_c = 0$. [Replace E_a by E_a/E_c for $a = a, b, c$, similarly for E and E .] By our hypotheses and the results of the first paragraph of the proof we may assume then that $E_d = E_a \oplus E_b$ and that $E_b \cup E_c = E_a \oplus E_b$.

Now as in the proof of the First Noether Isomorphism Theorem [2, p. 27], we have the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & E_a \oplus E_b & \longrightarrow & E_a \oplus E_b & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & E_b & \xrightarrow{\varphi_d} & E_a \oplus E_b & \xrightarrow{\varphi_d} & E_b \cup E_c \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & E_a \oplus E_b & & 0 \\
 & & & & \downarrow & & \downarrow \\
 0 & \longrightarrow & E_c/(E_a \oplus E_b) & \longrightarrow & 0 & \xrightarrow{h} & E_b \cup E_c \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

with solid rows and columns, where h is the unique morphism which makes the lower right hand square commutative: e.g., we may take $h = (\alpha_{h,d}, \beta_{h,d})$, where $\alpha_{h,d}: E_a \rightarrow E_b$ and $\beta_{h,d}: E_b \rightarrow E_b$. One verifies that the kernel of h is iso-

surjects to $K_a \cap K_b$. By the above diagram this kernel is also isomorphic to $K_{a'}/(K_a \oplus K_b)$. Since we have proved the lemma.

The following alternative formulation of the Box rule Lemma (3.1.3) may be of some interest.

(3.1.4) Proposition. Let K , P_1 , and P_2 be in $\mathcal{K}_n$. If $K \subset P_1 \cup P_2$, then

$$\frac{K}{(P_1 \cap K) \cup (P_2 \cap K)} = \frac{(P_1 \cup K) \cap (P_2 \cup K)}{(P_1 \cap P_2) \cup K}.$$

(3.1.5) Corollary. If everything is as in the above lemma, the induced sequence

$$0 \longrightarrow K_a \cup K_b \longrightarrow P_a \cup P_b \longrightarrow K_a \cup K_b \longrightarrow 0$$

is exact if and only if either $K_a \cup K_b = K_b$ or $K_a = K_a \cap K_b$.

Proof. The corollary is immediate from the Box rule lemma (3.1.3), the commutativity of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_a & \longrightarrow & P_a & \longrightarrow & K_a \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & K_a \cup K_b & \longrightarrow & P_a \cup P_b & \longrightarrow & K_a \cup K_b \longrightarrow 0, \end{array}$$

and the exactness of the top row of the diagram.

We complete the proof of 3.1.2 by induction on $|S|$. For this we need the following lemma, which will also be needed later in a different context.

(3.1.6) Lemma. Let S be a set of isomorphic ele-

space is a partially ordered set, 1 will be greatest and a least element. Assume that Theorem 3.1.2 holds for any set $\mathcal{P}^0$ of incomparable elements in any partially ordered set $\mathcal{P}^0$ with a greatest and a least element, such that $||\mathcal{P}^0|| < ||\mathcal{P}||$. Let $\{x_j: j \in J\}$ be such that $x_j \leq b$ in $\mathcal{P}$ such that

- (1) $\{x_j: j \in J^0\}$ is quasi-independent, and
- (2) $\{x_j: j \in J_0\}$ is quasi-independent.

Let

$$\mathcal{R}: 0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0,$$

in $\mathcal{R}[\mathcal{I}, \mathcal{R}]$, be an exact sequence, and let

$$\mathcal{R}^0: 0 \longrightarrow \mathcal{X}^0 \longrightarrow \mathcal{Y}^0 \longrightarrow \mathcal{Z}^0 \longrightarrow 0,$$

in $\mathcal{R}[\mathcal{I}^0, \mathcal{R}]$, be an exact sequence. Then if $x \in \mathcal{I}^0(\mathcal{Z})$, we have

$$(3.1.7) \quad \mathcal{X}_x^0 / \bigcup_J \mathcal{X}_J = \{ \bigcup_{J^0} \mathcal{X}_J \cap \mathcal{X}_0 \} / \bigcup_{J_0} \mathcal{X}_J.$$

[Proof]. Let $\mathcal{I}^0 = \mathcal{I}^1 + \{x^0\}$, $x \in \mathcal{I}^0(\mathcal{I}^1)$, and $\pi^0: \mathcal{R}[\mathcal{I}^1, \mathcal{R}] \longrightarrow \mathcal{R}[\mathcal{I}^0, \mathcal{R}]$ be the corresponding functor. By (1) and since $||\mathcal{I}^1\mathcal{P}|| < ||\mathcal{I}||$, we have by our hypothesis that Theorem 3.1.2 holds and hence that $\pi^0(\mathcal{R}^1)$, in $\mathcal{R}[\mathcal{I}^0, \mathcal{R}]$, is exact.

Similarly, let $\mathcal{I}^2 = \mathcal{I}^1 + \{x_0\}$, $x \in \mathcal{I}^2(\mathcal{I}^1)$, and $\pi^2: \mathcal{R}[\mathcal{I}^1, \mathcal{R}] \longrightarrow \mathcal{R}[\mathcal{I}^2, \mathcal{R}]$ be the corresponding functor. Then by (2) and since $||\mathcal{I}_0\mathcal{P}|| < ||\mathcal{I}||$, we have that $\pi^2\pi^0(\mathcal{R}^1)$, in $\mathcal{R}[\mathcal{I}^2, \mathcal{R}]$, is exact.

Now focusing our attention on x, x_0, x in $\mathcal{I}^2$, it is sufficient to verify that we have



from the exact sequence $H^2(X^0(x))$. Moreover, this diagram satisfies all of the hypotheses of the two Kato Lemmas (3.1.3). Hence we have 3.1.7, which completes the proof.

Proof of Theorem 3.1.6. Let us assume that we have established Theorem 3.1.6 for all X^0 (out of isomorphic classes of X^0 (see 3.1.6)) such that $|X^0| = |x|$. Now since $\{b_j : j \in J\}$ is quasi-independent, we have b is x such that

$$\bigcup_{j \in J} b_j \cap b_0 = \bigcup_{j \in J} b_j.$$

Moreover, we have (i) and (ii) of 3.1.6 from (i) and (ii) of 3.1.1. Thus we have from 3.1.7 (see 3.1.5) that $\bigcup_j b_j = b_0^x$, which gives us the surjectivity of $n(x)$.

(3.2.1) definition. As to the case of quasi-independence, we define inductively what we mean by quasi-independent. We will say that the family $\{b_j : j \in J\}$ is quasi-independent if $|J| = 1$, or if there is p in

α such that $\{B_j : j \in J'\alpha\}$ is nearly quasi-independent, and $\{B_j : j \in J_\alpha\}$ is quasi-independent.

[3.2.2]. *Lemma.* Let the exact sequence

$$R: 0 \longrightarrow X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z \longrightarrow 0$$

be in $\mathcal{B}[X, Y]$. If $\{B_j : j \in J'\}$ is nearly quasi-independent, then $\{B_j : j \in J\}$ is quasi-independent.

[Proof]. By hypothesis we have p in J such that

$\{B_j : j \in J_p\}$ is quasi-independent. Let $X^1 = X + (X_p)$, $z \in X^1/Z$, and $R: \mathcal{B}[X, R] \longrightarrow \mathcal{B}[X^1, R]$ be the corresponding function. By Theorem 3.1.2 we have the exact sequence

$$\mathcal{B}(X), 0 \longrightarrow \mathcal{B}(R) \longrightarrow \mathcal{B}(Y) \longrightarrow \mathcal{B}(Z) \longrightarrow 0$$

in $\mathcal{B}[X^1, R]$.

Now fix each j in J we have $B_j = \mathcal{B}(X)_j = \mathcal{B}(Y)_j + \mathcal{B}_j$.

Therefore we have

$$\bigcup_{j \in J'} B_j \cap B_p = \bigcup_{j \in J'} B_j \cap B_p = \bigcup_{j \in J'} B_j.$$

But $\mathcal{B}(R)_p \cap \bigcup_{j \in J'} B_j \cap B_p = \mathcal{B}(R)_p \cap \bigcup_{j \in J'} B_j \cap B_p \subset \mathcal{B}_p(B_p) = 0$.

Thus $\bigcup_{j \in J'} B_j \cap B_p \subset \text{Ker } \mathcal{B}(R)_p = \bigcup_{j \in J'} B_j$. Moreover, since

$0 \neq p$ and $X_p \subset \{X'\alpha\}$, we have $\bigcup_{j \in J} B_j = \bigcup_{j \in J'} B_j \cap B_p$. Thus

$$\bigcup_{j \in J} B_j = \bigcup_{j \in J'} B_j \cap B_p.$$

Now we finish the proof by induction on $|[J']|$. Note

because $\{B_j : j \in J'\alpha\}$ is nearly quasi-independent and

$|[J'\alpha]| < |[J']|$, $\{B_j : j \in J'\alpha\}$ is quasi-independent. It is

easy to see that quasi-independence implies nearly quasi-independence. Therefore since $|\{x_p\}| = |\{x\}|$ and $\{x_j : j \in I_p\}$ is quasi-independent, we have that $\{x_j : j \in I_p\}$ is quasi-independent. Thus $\{x_j : j \in J\}$ is quasi-independent.

(3.3) Now we give a condition on D which allows us effectively to determine $\text{hd } D$.

(3.3.1) Definition. We say that D is $\mathcal{H}(I, R)$, or basic if for each x in I , the family $\{x_j : j \in \text{supp}(\{x\}/x)\}$ is quasi-independent.

(3.3.2) Theorem. Let α denote the greatest element of I . Let $J = \text{supp}(\{\alpha\}/\alpha)$ and for each $j \in J$, let $F_j : (I, R) \rightarrow (I_j, R)$ be the restriction functor. Then if D is basic,

$$\text{hd } D = \sup_j \{ \text{hd}(\{D_{\alpha_j'} \cup_j D_j\}), \text{hd } F_j(D) \}.$$

The proof will be divided into the following sequence of lemmas.

(3.3.3) Lemma. Let the exact sequence

$$E : 0 \longrightarrow R \longrightarrow P \longrightarrow D \longrightarrow 0$$

be in $\mathcal{H}(I, R)$. If D is basic, then R is basic.

Proof. The proof follows immediately from 3.3.1 and Lemma 3.3.2.

The following lemma is rather obvious, but is included here in order to simplify the proofs of the four lemmas following it.

(3.3.4) Lemma. Let $\mathcal{B}$ and $\mathcal{B}'$ be abelian categories.

$\mathcal{B}$ and $\mathcal{B}'$ be subcategories of the class of objects of $\mathcal{B}$, and $F: \mathcal{B} \rightarrow \mathcal{B}'$ be a functor such that, if P is in $\mathcal{B}' \cap \mathcal{B}$ is projective in $\mathcal{B}$, then $F(P)$ is projective in $\mathcal{B}'$. Moreover, suppose that for any B in $\mathcal{B}$, there is a projective P in $\mathcal{B}$ and K in $\mathcal{B}$ such that the sequence

$$B \longrightarrow K \longrightarrow P \longrightarrow 0 \longrightarrow 0$$

and

$$F(B): 0 \longrightarrow F(K) \longrightarrow F(P) \longrightarrow F(0) \longrightarrow 0$$

are exact. Then,

(1) $\text{hd } B \leq \text{hd } F(B)$ for all B in $\mathcal{B}$.

(2) If B is projective whenever B is in $\mathcal{B}$ and $F(B)$ is projective, then $\text{hd } B = \text{hd } F(B)$, for B in $\mathcal{B}$.

Proof. We prove (1) and (2) simultaneously by induction on $\text{hd } B$. If $\text{hd } B$ is projective, then $F(B)$ is projective. We assume that both (1) and (2) hold whenever $\text{hd } B' < \text{hd } B \neq 0$, and that B' satisfies the hypothesis of (1) and (2). If $\text{hd } F(B) = 0$, then we are finished. If not, then we have from the exact sequences π and $F(\pi)$ that $\text{hd } K = \text{hd } B - 1$ and $\text{hd } F(K) = \text{hd } F(B) - 1$, respectively. Moreover, we have that K is in $\mathcal{B}$ and thus applying the inductive hypothesis $\text{hd } K \leq \text{hd } F(K)$. If we assume the hypothesis of (2), then we have $\text{hd } K = \text{hd } F(K)$ and $F(K)$ is not projective if B is not projective, which is sufficient to establish that $\text{hd } F(B) = \text{hd } B$.

(3.3.5) Lemma. If B is in $\mathcal{B}(I, R)$ and $i < j \in I$, then
 $\text{hd } B = B_j/B_i$.

Proof. Define the functor $F: \mathcal{B}(I, R) \rightarrow \mathcal{B}$ by $F(x) = B_j/B_i$. The proof follows from 3.3.4, taking $\mathcal{B} = \mathcal{B}(I, R)$, after routinely verifying that F preserves all projectives and all short exact sequences.

(3.3.6) Lemma. Let B in $\mathcal{B}(I, R)$ be basic, $B_\infty = \bigcup_j B_j$, where n is the greatest element of I , and $\mathcal{B} = \text{cls } \{[n]^\perp B_n\}$. Let $k = [n]$ and let $F: [I, R] \rightarrow [I, R]$ be the restriction functor. Then $\text{hd } F(B) = \text{hd } B$.

Proof. We shall use 3.3.4. Let us denote by $\mathcal{B}$ all B in $[I, R]$ which satisfy the hypothesis of the lemma.

We see from Theorem 3.1.2 that we may find a projective P in $\mathcal{B}$ such that

$$B \rightarrow P \rightarrow B \rightarrow P \rightarrow B \rightarrow P \rightarrow \cdots$$

is an exact sequence. We have by 3.3.3 that B is basic, and it follows from Theorem 3.1.2 that $B_\infty = \bigcup_j B_j$. Since B is in $\mathcal{B}$, since F is a restriction functor, it preserves all short exact sequences, and one easily verifies that F preserves projectives. It remains to be shown that for B in $\mathcal{B}$, if $F(B)$ is projective, then B is projective.

We suppose $F(B)$ is projective. Then by 3.4 we must have an isomorphism

$$\text{dr } \bigoplus_{i \in I} E_i(F_i) \xrightarrow{\sim} F(p),$$

where each F_i is projective. The morphism ϕ determines

uniquely the morphism

$$d': \bigoplus_{j=1}^n d_j^{-1}(p_j) \longrightarrow D,$$

where $d_j^{-1} = d_j$ for $j \leq n$. It follows that

$$0 \longrightarrow E \longrightarrow F \longrightarrow D \longrightarrow 0$$

is exact, where

$$F = \bigoplus_{j=1}^n d_j^{-1}(p_j).$$

From the first part of the proof we have $E_n = \bigcup_j E_j$. But since $E_j = \ker d_j = 0$ for all j in n , we have $0 = E_n = \ker d'_n$, which shows that d' is an isomorphism. Thus by 1.4, D is projective. Taking $D = 0$, we may now apply 3.3.4 to get $\ker D = \ker F(0)$.

(3.3.7) Lemma. Let D in $\mathcal{M}(I, R)$ be basic, $d = \min \{[x]'\alpha\}$, where α is the greatest element of I , and let D' in $\mathcal{M}(I, R)$ be defined by $D'_\alpha = \bigcup_j D_j$ and $D'_j = D_j$ for $j < \alpha$. Then $\ker D = \ker D'$.

Proof. Define $\alpha: \mathcal{M}(I, R) \rightarrow \mathcal{M}(I, R)$ by $\alpha(x) = D'$, where D' is as in the statement of the lemma. We easily verify that α preserves projectives. Let

$$0 \longrightarrow E \longrightarrow F \longrightarrow D \longrightarrow 0$$

be an exact sequence with F projective (implying that E is in $\mathcal{M}(I, R)$) and D basic. A simple application of Theorem 3.1.2 establishes that

$$\alpha(E) \subset 0 \longrightarrow \alpha(E) \longrightarrow \alpha(F) \longrightarrow \alpha(D) \longrightarrow 0$$

is exact. It follows from the above and from 3.3.3 that 3.3.4 may be applied to obtain $\text{hd } D = \text{hd } D'$.

(3.3.5) Lemma. Let $X^1 = X + \{x\}$, where $x \in \text{cls } \{[u] \mid u\}$ and u is the previous element of X . And let R be the functor defined in 3.3.4. If p is basic, then $\text{hd } D = \text{hd } R(p)$.

PROOF. We apply 3.3.4 by letting $\mathcal{B} = \mathcal{B}$ be the class of all basic diagrams in $\mathcal{B}(E, W)$. We see from 3.3.4 that if D is projective, then $R(p)$ is projective. And if $R(p)$ is projective, then by noting that $D = \text{Pr}(D)$ (where $\text{Pr}: X^1(E) \rightarrow X(E)$ is the natural map) and that $\text{Pr}(X_1^{(1)}(x_1))$ is projective for all 1 in X^1 such that p_1 is projective, we see that D is projective. Moreover, for p in $\mathcal{B}$, we have a projective P (necessarily in $\mathcal{B}$) such that

$$\text{hd } D \rightarrow D \rightarrow P \rightarrow R \rightarrow D,$$

in $\mathcal{B}(E, W)$, is exact and by 3.3.3, R is basic, i.e., R is in $\mathcal{B}$. Also, we have by 3.3.2 that

$$R(p): D \rightarrow R(p) \rightarrow R(P) \rightarrow R(D) \rightarrow D$$

is exact. Then by 3.3.4, $\text{hd } D = \text{hd } R(p)$.

PROOF OF THEOREM 3.3.2. Letting p' be as in 3.3.2, we have the exact sequence

$$E, D \rightarrow p' \rightarrow D \rightarrow R_p(R_p' \cup_j R_j) \rightarrow D,$$

from which it follows that

$$\{V\} \quad \text{hd } D \neq \text{sup} [\text{hd } D', \text{hd } (R_p' \cup_j R_j)]$$

Now we may apply 3.3.7 to get $\text{hd } \mathfrak{B} \leq \text{hd } \mathfrak{B}'$, and 3.3.6 and 3.3.5 to get

$$\text{hd } \mathfrak{B} = \text{hd } \mathfrak{B}(\mathfrak{B}) \leq \text{hd } (\mathfrak{B}_M' \cup \mathfrak{B}_J),$$

which shows that (1) is an equality. But by 3.3.6 and 3.6.7

$$\text{hd } \mathfrak{B}' = \text{hd } \mathfrak{F}(\mathfrak{B}) = \sup_J \text{hd } \mathfrak{F}_J(\mathfrak{B}).$$

From which it follows that

$$\text{hd } \mathfrak{B} = \sup_J \{ \text{hd } (\mathfrak{B}_M' \cup \mathfrak{B}_J), \text{hd } \mathfrak{F}_J(\mathfrak{B}) \}.$$

(3.3.3) Corollary. If $\mathfrak{B}$ is $\mathfrak{B}(I, \mathfrak{B})$ is basic, then $\text{hd } \mathfrak{B} = \text{ql-dim } \mathfrak{B}$.

Proof. The proof is by induction on $|I|$. Thus we have, since $\mathfrak{F}_j(\mathfrak{B})$ is basic for all j in J , that $\text{hd } \mathfrak{F}_j(\mathfrak{B}) \leq \text{ql-dim } \mathfrak{B}$. Obviously $\text{hd } (\mathfrak{B}_M' \cup \mathfrak{B}_J) \leq \text{ql-dim } \mathfrak{B}$, hence by Theorem 3.3.2, $\text{hd } \mathfrak{B} \leq \text{ql-dim } \mathfrak{B}$.

(3.4) In this section, we consider the case in which $\mathfrak{B}$ is not basic.

(3.4.1) Definition. Let α be the greatest element of I , $J = \text{supp } ([\alpha]|\alpha)$, and $\mathfrak{B}$ be an $\mathfrak{B}(I, \mathfrak{B})$. We say that $\mathfrak{B}$ is **strongly basic** if

- (1) there is p in J such that the families $\{\alpha_j : j \in J' \setminus p\}$ and $\{\alpha_j^{-1} : j \in I_p\}$ are quasi-independent, and
- (2) for any x in $I' \setminus \alpha$, $\{\alpha_j : j \in \text{supp } ([x]|\alpha)\}$ is quasi-independent.

(3.4.2) Theorem. With the notation of 3.4.1, let $\mathfrak{B}$

is merely basic. Let B' in $\mathcal{B}(L, R)$ be defined by $B'_0 = \bigcup_j B_j$ and $B'_j = B_j$ for $j \in \mathbb{N}$. Then

$$\text{hd } B' \geq \text{hd } \left\{ \left(\bigcup_{j \in \mathbb{N}} B_j \right) / \bigcup_{j \in \mathbb{N}} B_j \right\} = 1.$$

Moreover, if

$$\text{hd } \left(B_0 / \bigcup_{j \in \mathbb{N}} B_j \right) \neq \text{hd } \left\{ \left(\bigcup_{j \in \mathbb{N}} B_j \right) / \bigcup_{j \in \mathbb{N}} B_j \right\} = 1.$$

Then $\text{hd } B \neq \text{hd } B'$.

Lemma. Throughout this proof, we shall let $\mathcal{B}$ denote $\left\{ \left(\bigcup_{j \in \mathbb{N}} B_j \right) / \bigcup_{j \in \mathbb{N}} B_j \right\}$. If $\mathcal{B} \neq \emptyset$, then since B is merely basic, $\mathcal{B}$ is also merely basic and hence $\mathcal{B}'$ is basic. It follows immediately from 3.3.7 that $\text{hd } B \geq \text{hd } B' \geq 0 = \text{hd } \mathcal{B} + 1$. (We use the convention that $\text{hd } \emptyset = -1$.) So let us assume that $\mathcal{B} \neq \emptyset$, and consider the exact sequence

$$B_0 \hookrightarrow B \longrightarrow B' \longrightarrow \mathcal{B} \longrightarrow 0$$

and

$$B'_0 \hookrightarrow B' \longrightarrow B'' \longrightarrow \mathcal{B}_0 \longrightarrow 0,$$

where F is projective and F' is defined by $F'_0 = \bigcup_j F_j$ and $F'_j = F_j$ for $j \in \mathbb{N}$. Since F is projective, it follows that F' is projective. Further, since $\mathcal{B} \neq \emptyset$, $\mathcal{B}$ and $\mathcal{B}'$ are not projective, and we have from 3 and 3.1,

$$(1) \quad \text{hd } B = \text{hd } B' = 1,$$

and

$$(2) \quad \text{hd } B'' = \text{hd } B' = 1.$$

Now consider $\mathcal{B}$ and Lemma 3.1.5. The family

$\{B_j : j \in \mathbb{N}\}$ clearly satisfies hypotheses (1) and (2) of

Lemma 3.1.6) that

$$(3) \quad R_M' \cup_I R_2 = R.$$

Moreover, $\{R_j : j \in J\}$ is nearly quasi-independent, and hence by 3.8.7, $\{R_j : j \in J\}$ is quasi-independent. It follows that R and R' are birational, thus by Theorem 3.3.6

$$(4) \quad \text{hd } R = \sup_j \{ \text{hd } (R_M' \cup_I R_j), \text{hd } P_j(R) \},$$

and

$$(5) \quad \text{hd } R' = \sup_j \{ \text{hd } (R_M' \cup_I R_j), \text{hd } P_j(R') \}$$

$\{P_j(R) = P_j(R')\}$. From (2), (3), and (5) it follows that $\text{hd } R' \geq \text{hd } R + 1$, which we to be shown.

To complete the proof, consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_M & \longrightarrow & P_M & \longrightarrow & R_M & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & R_M' & \longrightarrow & \bigcup_I P_j & \longrightarrow & \bigcup_I R_j & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

with exact rows and columns. By the 3 lemma, we have the exact sequence

$$(6) \quad 0 \longrightarrow R_M/R_M' \longrightarrow P_M' \cup_I P_j \longrightarrow R_M' \cup_I R_j \longrightarrow 0,$$

now if $R_M' \cup_I R_j$ is projective, then $\bigcup_I R_j$ is a direct summand of R . From this it follows that $\text{hd } R = \text{hd } R'$, which we to be shown. So let us assume that $R_M' \cup_I R_j$ is not projective. Then we have from (6), since $P_M' \cup_I P_j$ is projective,

live, that

$$(7) \quad \text{hd } (K_M/K_M^c) + 1 = \text{hd } (K_M' \cup_{\mathcal{J}} K_J).$$

We now finish the proof by reduction ad absurdum. So suppose that $\text{hd } \mathcal{B} < \text{hd } \mathcal{B}'$. Then by (1) and (2), $\text{hd } \mathcal{B} < \text{hd } K'$. From this and (3) and (5),

$$\text{hd } (K_M' \cup_{\mathcal{J}} K_J) < \text{hd } (K_M' \cup_{\mathcal{J}} K_J).$$

However, we have the exact sequence

$$0 \longrightarrow K_M' \cup_{\mathcal{J}} K_J \longrightarrow K_M' \cup_{\mathcal{J}} K_J \longrightarrow K_M'/K_M^c \longrightarrow 0,$$

which by 3.3.4 gives us

$$(8) \quad \text{hd } (K_M'/K_M^c) = \text{hd } (K_M' \cup_{\mathcal{J}} K_J) + 1$$

It follows from (7), (8), and (3) that

$$\text{hd } (K_M' \cup_{\mathcal{J}} K_J) = \text{hd } \mathcal{B} + 2,$$

which contradicts the hypothesis of the theorem.

(3.3.3) Lemma. With the notation of 3.3.1 and the proof, if $\text{hd } \mathcal{B} = \text{ql_dim } \mathcal{B}$, then $\text{hd } \mathcal{B} \geq 1 + \text{ql_dim } \mathcal{B}$.

Proof. Clearly, if $\text{hd } \mathcal{B} = \text{ql_dim } \mathcal{B}$, then

$$\text{hd } (K_M' \cup_{\mathcal{J}} K_J) \neq \text{hd } \mathcal{B} + 2.$$

Thus by 3.3.2, $\text{hd } \mathcal{B} \geq \text{hd } \mathcal{B} + 1$, which completes the proof.

(3.3.4) Let $\mathcal{F}$ be a set of inseparable elements of $\mathcal{E}$ and let x in $\mathcal{E}$ be such that $[x] \in \mathcal{A}$. Let $\mathcal{E}^{\mathcal{F}} = [x] \cup \{x\}$. We denote by $P_{\mathcal{F}}$ the restriction function from $[\mathcal{E}, \mathcal{B}]$ to $[\mathcal{E}^{\mathcal{F}}, \mathcal{B}]$.

(3.3.5) theorem. If D in $\mathcal{M}(I, \mathbb{R})$ is not basic, then there is a set J of incompatible elements of I such that

- (1) $\mathcal{F}_\pi(D)$ is nearly basic but not basic, and
 (2) $\text{hd } D \neq \text{hd } \mathcal{F}_\pi(D)$

for all π in I such that $\{x\} \subseteq \pi$.

Proof. Since D is not basic, there is some set J' of incompatible elements such that $\{D_j : j \in J'\}$ is not quasi-independent. Consider

$$E = \{D_j : J' \subseteq I, J' \text{ is a set of incompatible elements, and } \{D_j : j \in J'\} \text{ is not quasi-independent}\}.$$

Clearly E is a finite partially ordered set under set inclusion, and hence there is a maximal element of E , say J . We now need only show that J satisfies the conclusions of the theorem.

First, it is clear that $\mathcal{F}_\pi(D)$ is nearly basic. Next we shall establish that $\text{hd } D \neq \text{hd } \mathcal{F}_\pi(D)$ by 3.3.4. But since $\mathcal{F}_\pi$ is a translation function, it preserves all short exact sequences. So we need show only that $\mathcal{F}_\pi$ preserves projectives, which will be established if we show that $\mathcal{F}_\pi(\mathcal{P}_1^T(\mathcal{P}_1))$, where $\mathcal{P}_1$ is π -projective, is projective (here we assume π is projective). For i in $\mathbb{R}$, it is clear that $\mathcal{F}_\pi(\mathcal{P}_1^T(\mathcal{P}_1)) = \mathcal{P}_1^{\pi \circ i}(\mathcal{P}_1)$. If $i \notin \mathbb{R}$, then $\pi(i) \cap \mathbb{R} = \{r\}$ or $\pi(i) \cap \mathbb{R} = \emptyset$. Thus $\mathcal{F}_\pi(\mathcal{P}_1^T(\mathcal{P}_1))$ is equal to $\mathcal{P}_1^{\pi \circ i}(\mathcal{P}_1)$ or 0, respectively. Hence by 3.3.4, $\text{hd } D \neq \text{hd } \mathcal{F}_\pi(D)$, which was to be shown.

(3.5) In 2.3.2, we noted that if p was the smallest

integer such that $\text{hd } K^p = \text{hd } K$, then $\text{hd } H_1(K) = \text{hd } K + p + 1$. We should like to use the results of Chapter III to determine p . We have the following

(3.3.1) Lemma. Let the exact sequence of 3.3 be given. Then the smallest p such that $\text{hd } K^p = \text{hd } K$ is the smallest p such that K^p is basic.

We point out that the lemma would follow if K^p satisfied the hypotheses of the following proposition.

(3.3.2) Proposition. Let I be a set of independent elements of I and let x in I satisfy $[x] = 0$. For σ in $S(I, I)$, we denote $\bigcup_{j \in \sigma} D_j$ by $W(x, \sigma)$, and for b in σ , we denote $\bigcup_{j \in \sigma} (D_j \cap D_b) / \bigcup_{j \in \sigma} D_j$ by $W(b, \sigma)$.

(3.3.3) Proposition. Let $D \neq 0$ be basic and suppose that for all x in I and σ in $S(I, I)$, either $W(x, \sigma) = 0$ or $\text{hd } W(x, \sigma) = n$. Then $\text{hd } D = n$.

Proof. The proof may be accomplished by a straightforward argument and Theorem 3.3.2.

(3.3.4) Proposition. Suppose x in $S(I, I)$ is not basic and that there is a set J of independent elements such that J is minimal with respect to being non-quasi-independent (see 3.4.3). Furthermore, suppose there is x in I such that $[x] = x$ and there is b in J such that $\text{hd } W(b, J) = n + 1$, and either $\text{hd } W(x, J) = n$ or $n + 1$. Then $\text{hd } D \geq n + 1$.

Proof. Let F_x be the functor defined for x and J as

3.4.4. Then $P_E[X]$ is evenly basic, but not basic. Hence by the proof of 3.4.5, $\text{hd } \pi \leq \text{hd } P_E[X]$. Moreover, it is clear that $\text{hd } \pi(\langle \pi, X \rangle) \neq \text{hd } \pi(\langle \pi, X \rangle) + 2$, so we have by 3.4.2 that $\text{hd } P_E[X] = \text{hd } \pi(\langle \pi, X \rangle) + 1 = n + 1$, which completes the proof.

(3.5.5) Conjecture. In all the examples we have studied such that X^P (as in 2.3) is basic, X^P has satisfied the hypothesis of 3.5.3. We conjecture that this is always true.

Furthermore, recall that if X^P is not basic, then by 3.4.5 we have π and β such that β is minimal in the sense of 3.4.5. Again, in all the examples considered, we have found that $\text{hd } \pi(\langle \pi, X \rangle) = \text{hd } \beta$ for all X^P as in 2.3. It seems reasonable to believe that this is always true.

We note that Conjecture 3.5.1 follows from Conjecture 3.5.5, using Propositions 3.5.3 and 3.5.4.

CHAPTER IV

PARTIALLY ORDERED SETS OF DIMENSION TWO

(4.1) As before, we assume that L is a partially ordered set with a greatest and a least element.

(4.1.1) Definition. For a set of incomparable elements J , let us define by induction on $|J|$ what it means for J to be decision-2 free. If $|J| = 1$, then J is decision-2 free. Suppose that we have defined decision-2 free for a set J' of incomparable elements such that $|J'| < |J|$. Then J is decision-2 free if there is p in J so that $J' \cup p$ is decision-2 free, and if there is q in J so that

$$L(p) \cap [q] = \{q\}.$$

(4.1.2) Definition. Let us say that L is decision-2 free if for any x in L , $J = \max\{[x] \setminus x\}$ is decision-2 free.

(4.1.3) Theorem. If L is decision-2 free, then $\text{gl.dim } [L, x] + 2 = \text{gl.dim } x$.

Proof. By Theorem 3.1.2 it suffices to show that $\text{hd } K_x(x) = 2 + \text{gl.dim } x$ for any x in x and x is 1. Using the notation in 2.3, we have the exact sequences

$$(1) \quad 0 \longrightarrow K^0 \longrightarrow K^1 \longrightarrow K_2(x) \longrightarrow 0$$

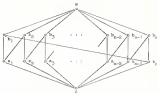
and

$$(2) \quad 0 \longrightarrow K^1 \longrightarrow K^2 \longrightarrow K^3 \longrightarrow 0.$$

We show that $\text{hd } h_1[h] = r + q_1 \dim U$, it suffices by 2.3.2 and 2.3.3 to show that $\text{hd } h^1 = q_1 \dim U$. And this follows immediately from 3.3.5 once we have shown that h^1 is basic. To do this, let $x \in U$ and $J = \max \{|x|/q\}$. We must show that $\{h_{j+1}^1 : j \in J\}$ is quasi-independent. But it suffices by 3.3.2 to show that $\{h_j^0 : j \in J\}$ is nearly quasi-independent. We may, in fact, show that if U is any element of $\mathcal{B}(x, R)$ and J is any decision- R free subset of U , then $\{h_j^0 : j \in J\}$ is nearly quasi-independent. We show this by induction on $|x|$.

First assume that for J' such that $|J'| < |J|$, we have shown that if J' is decision- R free, then $\{h_{j'}^0 : j' \in J'\}$ is nearly quasi-independent. Since J is decision- R free, there is p in J so that $J \setminus p$ is decision- R free, and there is q in J so that $\{q\} = [J'q] \cap \{p\}$. By the inductive hypothesis, $\{h_{j'}^0 : j' \in J'q\}$ is nearly quasi-independent, and clearly $\{h_q^0\}$ is quasi-independent. This completes the proof.

(N.P.1) The partially ordered set defined for $x \in \mathcal{B}$ by the graph (directed upward)



will be called a suspended crown, the name given to it by the diamondator, Barry Mitchell.

We have the following

[3,2.2] Theorem. If R is a suspended crown, then $gl_dim [R, R] = 3 + gl_dim R$.

Proof. We first note that $gl_dim [R, R] \leq 3 + gl_dim R$ by [1, lemma 3.5].

Let k be R be such that $hd\ k = gl_dim R$. Using the notation in 2.3, we have the exact sequence

$$(1) \quad 0 \longrightarrow k^0 \longrightarrow k^1 \longrightarrow k_2[k] \longrightarrow 0$$

and

$$(2) \quad 0 \longrightarrow k^1 \longrightarrow k^2 \longrightarrow k^3 \longrightarrow 0.$$

To show that $hd\ k_2[k] \geq hd\ k + 3$, it suffices to show by 2.3.2 that $hd\ k^1 \geq hd\ k + 1$. To do this we use 2.3.3. (We

could, in $H^1(\mathbb{R}^n)$ ($n \geq 3, 4, 5$).

To see (3.4.3), we must show that π^1 is really basic.

First we note that $(x_{k_1}^2 : 1 \leq k_1 \leq n-1)$ is easily proved to be quasi-independent. Hence we have that $(x_{k_1}^1 : 1 \leq k_1 \leq n-1)$ is quasi-independent by (3.2.2). From (3.1.2) we get the exact sequence

$$0 \longrightarrow \bigcup_{i=1}^{n-1} x_{k_1}^1 \longrightarrow \bigcup_{i=1}^{n-1} x_{k_1}^1 \longrightarrow \bigcup_{i=1}^{n-1} x_{k_1}^2 \longrightarrow 0.$$

But $\bigcup_{i=1}^{n-1} x_{k_1}^2 = A = x_{k_1}^2$ and $x_{k_1}^1 = \bigcup_{i=1}^{n-1} x_{k_1}^1$ since $\bigcup_{i=1}^{n-1} x_{k_1}^1 \subset \bigcup_{i=1}^{n-1} x_{k_1}^1$

and $x_{k_1}^1 \subset x_{k_1}^1$. Therefore, $x_{k_1}^1 = \bigcup_{i=1}^{n-1} x_{k_1}^1 \subset \bigcup_{i=1}^{n-1} x_{k_1}^1 = x_{k_1}^1$. Thus

$\bigcup_{i=1}^{n-1} x_{k_1}^1$ is $x_{k_1}^1$, the kernel of $x_{k_1}^1 \longrightarrow x_{k_1}^2$ and hence

$$x_{k_1}^1 \cap \bigcup_{i=1}^{n-1} x_{k_1}^1 = x_{k_1}^1 \cap x_{k_1}^1 = x_{k_1}^1.$$

Since $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : A \longrightarrow A \oplus A = x_{k_1}^1$ is a kernel for $x_{k_1}^1$, we

must have $x_{k_1}^1 \subset A$. From (3.4.2) we have $\text{hd } K^1 \leq \text{hd } A + 1$,

which completes the proof.

(4.3) In this section, we will prove that if I is not

decision-2 free, then I contains a suspended curve.

(4.2.1) Suppose that I is not decision-2 free. Then there is x in I so that $\max\{|x|/x\}$ is not decision-2 free. Therefore there is a minimal subset of $\max\{|x|/x\}$, say J , which is not decision-2 free. Let us continue to denote $\max\{|x|_J\}/J$ by I_J . Then for all j in J , we have j'_j is decision-2 free by the minimality of J , and $|I_J| = 2$, since J is not decision-2 free.

(4.2.2) Let J be any set of incompatible elements of I . If $j_1 \neq j_2 \in J$, $k \in I_{j_1}$, and $k \in I_{j_2}$, then we write the ordered triple $j_1 k j_2$. If we have a sequence of such triples

$$j_1 k_1 j_2, j_2 k_2 j_3, \dots, j_n k_n j_1,$$

then we abbreviate it to

$$j_1 k_1 j_2 k_2 j_3 \dots j_n k_n j_1.$$

It is clear that if $k \in I_{j_1}$, then there is $j_2 \in J(j_1)$ such that $j_1 k j_2$.

(4.2.3) Given $j_1 k_1 j_2$, there is $k_2 \neq k_1 \in I_{j_2}$ such that $j_2 k_2 j_3$, and either $k_1 < k_2$ or k_1 and k_2 are incompatible.

Proof. $j_1 k_1 j_2$ implies that $k_1 \in (J(j_2) \cap I_{j_1})$.

Therefore,

if $k \in I_{j_2}$, then $k < k_1$ or k and k_1 are incompatible.

In the latter case, we are done. And in the former case, we have $|I_{j_2}| = 2$, and hence there is at least one $k \in I_{j_2}$ so that $k > k_1$. Denote this element by k_2 .

By repeatedly applying 4.3.3, we may obtain for any $n \geq 2$

$$j_1 k_1 j_2 k_2 \cdots k_n j_n,$$

where, j is finite, and hence some two of the j 's must be the same. We finish without loss of generality ($n \geq 2$)

$$(4.3.4) \quad j_1 k_1 j_2 \cdots j_n k_n j_1,$$

where all of the j 's are distinct.

(4.3.5) Notice that 4.3.4 has the property that for all $1 \leq i \leq n-1$, either $k_i = k_{i+1}$ or k_i and k_{i+1} are incomparable. We call such a sequence a preloop. By 4.3.4, we mean a preloop such that for all $1 \leq i \leq n-1$, k_i and k_{i+1} are comparable.

(4.3.6) Every preloop "contains" a loop.

Proof. By "contains" we mean that the j 's of the loop are contained in the j 's of the preloop.

Let us consider a preloop with two j 's, say $j_1 k_1 j_2 k_2 j_1$. If it is not a loop, then we have $k_1 \neq k_2$. But $j_2 k_2 j_1$ implies $k_2 = k_1 \in I_{j_1}$, which is impossible.

Now let $n \geq n \geq 3$, and suppose that we have shown 4.3.6 to be true for all preloops with n j 's. Consider 4.3.4. Either 4.3.4 is a loop, or for some $1 \leq i \leq n-1$,

$$(I) \quad k_i = k_{i+1}$$

then $k_i = k_{i+1} = j_{i+2}$ ($j_{i+1} = j_1$), and hence $j_1 k_1 j_2 k_2 \cdots$ If

$$(II) \quad j_1 k_1 j_2 \cdots j_i k_i j_{i+2} \cdots j_n k_n j_1$$

is a preloop, then we may apply our inductive hypothesis and finish the proof. If (P) is not a preloop, then we must have by (1) that $k_{1+2} \neq k_1 \neq k_{1+1}$, which conflicts with the fact that k, j, k is a preloop. Thus we are finished.

(4.3.7) By a minimal loop we mean a loop for which no proper subset of the j 's forms a loop.

(4.3.8) If 4.3.4 is a minimal loop, then so is

$$(1) \quad j_1 k_n j_{n+1} \cdots j_{n+1} k_{n+1} j_1$$

for any $1 \leq n \leq (j_{n+1} \neq j_1)$.

Proof:- It is necessary and sufficient to show that k_1 and k_n are inseparable. For $n = 2$, this is trivial; so let $n \geq 3$. Since we have $j_n k_n j_1 k_1 j_n$, we have since k_n and k_1 are inseparable as $k_n \neq k_1$. But $k_n \neq k_1 \neq j_2$. Thus we get $j_n k_n j_2$, and thence the loop $(n \geq 3)$

$$j_2 k_2 j_3 \cdots j_n k_n j_2,$$

which is shorter than 4.3.4. That (1) is minimal is obvious.

(4.3.9) Let 4.3.4 be a minimal loop. Then $k_1 \neq j_1$ if and only if $1 = i$ or $1 = i+1$. [$(j_{n+1} \neq j_1)$].

Proof:- By 4.3.8 it suffices to show that if $k_1 \neq j_1$, then $1 = i$ or $1 = i+1$. Moreover, since 4.3.9 is trivial if $n = 2$, let $n \geq 3$. If $1 = n$, then we have $j_1 k_1 j_n k_n j_1$, which is a shorter loop than 4.3.4, since k_1 and k_n are inseparable by the proof of 4.3.8, a contradiction.

Without loss of generality, let $2 \leq 1 \leq n$ be minimal

such that $k_1 \neq k_2$. Then we have

$$(1) \quad j_1 k_1 j_1^{-1} \cdots j_n k_n j_n^{-1}$$

has either (1) is a shorter loop than $k_1 j_1 k_1^{-1}$, or $k_1 = k_{j_1}$. In the former case we have a contradiction, and in the latter we have $k_1 \neq k_{j_1} = j_{j_1+1}$, which contradicts the maximality of j .

(4.3.10) Let $k_1 j_1 k_1^{-1}$ be a minimal loop, a be an element of Γ preceding all of the k 's, and let x be as given in 4.3.1. If $B = \{a, a, j_1, k_1 \mid i = 1, \dots, n\}$, then B is a non-pended crown. Furthermore, let

$$B' = B \cup \{a \mid a \neq a \text{ and } [a] \cap B = \{a\}\}.$$

Then B is separated in B' , which is normal in Γ .

Proof. To show that B is a separated crown, it is sufficient to show that the j 's are incomparable, the k 's are incomparable, and k_1 is followed by exactly j_1 and j_{j_1+1} ($j_{j_1+1} = j_1$). We have already seen that the j 's are incomparable, and 4.3.9 demonstrates that k_1 is followed by exactly j_1 and j_{j_1+1} . We suppose $k_2 \neq k_1$, and without loss of generality that $i \neq j$. Then $k_2 \neq k_1 \neq j_1, j_{j_1+1}$ implies by the preceding that $j_2 = j_1$ and that $j_{j_2+1} = j_{j_1+1}$, which, as before, implies that $i = j$ since the j 's are distinct.

We now show that B' is a normal subset of Γ by showing that

$$(1) \quad j_1 a [B' \cap \langle a, a \rangle] = 1$$

for all $a \in a(B')$ and $a \in B' \setminus \{a \neq a\}$. However, $a \in B'$

implies $x \notin a$, so $x \in r$, and if $x \in \bar{B}$, then (1) clearly holds. Hence let us assume that $x \notin B'$. Now $x \in a(B') = a(B)$, and therefore there is $y \in B$ such that $x \in y$. Thus $\text{int}([a] \cap B)$ is not empty. We shall use this fact to divide the remainder of the proof into four cases.

Case I. Suppose $x \in \text{int}([a] \cap B)$. Then $x \in a \cap r$, and therefore $x \in B$, a contradiction.

Case II. Suppose $y_1 \in \text{int}([a] \cap B)$. Then $y_1 \in a \cap r$, and since $y_1 \in \text{int}([r]/a)$, we have $x \in r \in B$, a contradiction.

Case III. Suppose $y_2 \in \text{int}([a] \cap B)$. Then let $w_1, w_2 \in \text{int}(B' \cap (a, x))$. Since $y_2 \in a \in w_1, w_2$, we have from the definition of B' that $w_1, w_2 \in B$. It follows from 4.3.3 that $y_2 \in a \cap y_1, y_2$, which implies that $x = y_2 \in B$, a contradiction.

Case IV. Suppose $x \in \text{int}([a] \cap B)$. It follows that $x \in B'$, a contradiction.

It is trivial to verify that B is a separated subset of B' .

(4.3.11) **Lemma.** If I is not decision-2 free, then there is a separated subset B and a subset B' of I such that B is a separated subset of B' and B' is a normal subset of I .

(4.3.12) **Theorem.** $\text{gl-dim}(I, B) \leq 2 + \text{gl-dim } B$ if and only if I is decision-2 free.

Proof. We have by 4.3.3 that if I is decision-2 free,

then $gl_dim [I, R] = 2 + gl_dim R$. If I is not division-free, then by 4.3.11 we have a maximal chain R which is a repeated subset of I' . Hence by 1.6.4 and 4.3.8

$$gl_dim [R', R] = 1 + gl_dim R.$$

Moreover, since R' is a maximal subset of I ,

$$gl_dim [I, R] = 1 + gl_dim R,$$

by 1.6.8 and the above.

Mitchell [8, p. 226] defined what it means for any finite partially ordered set to be division free. In terms of sets with a least and a greatest element, I is said to be division free if it is totally ordered. Moreover, a finite partially ordered set I is said to be distributive if no two elements have a joined

From [8, p. 227] we have

(4.3.13) Theorem. The set I is division free if and only if

$$gl_dim [I, R] = 1 + gl_dim R,$$

If I is nondistributive, then equality holds,

Hence we have from 4.3.13,

(4.3.14) Theorem. The set I is division-free, but not division free if and only if

$$gl_dim [I, R] = 2 + gl_dim R,$$

The reader may compare 4.3.14 with the following result from Mitchell.

A finite partially ordered set L (not necessarily with a greatest and a least element) is said to "contain" a suspended crown if there is some ordered subset of it, say L_0 , such that either L is a suspended crown with $n \leq 3$ (see 4.3.1), or L is a suspended crown with $n \geq 3$, and there does not exist x in L such that $x_1, x_2 \leq x \leq y_1, y_2$.

(4.3.15) Theorem. (Mitchell) The set L is not decomposition free and does not contain a suspended crown if and only if

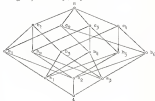
$$qd \dim [L, R] = 2 + qd \dim R,$$

CHAPTER V

3. DIMENSIONALITY

(3.1.1) In [2, p. 241], the question is asked whether $\text{gl dim } [1, \mathbb{R}] = \text{gl dim } \mathbb{R}$ is constant for all abelian subquivers $\mathbb{R}$ with projections and injections. If it is, then the unique number equal to $\text{gl dim } [1, \mathbb{R}] = \text{gl dim } \mathbb{R}$ is said to be the dimension of I . In the following, we give an example of a partially ordered set I such that I does not have a dimension.

(3.1.2) Let I be given by



Furthermore, let $\mathcal{B}_2$ and $\mathcal{B}$ be categories of vector spaces over fields of characteristic equal to two and not equal to two, respectively. We now show that

$$(3.1.3) \quad \text{Theorem: } \text{qd-dim } [2, \mathcal{B}_2] = 4, \text{ and} \\ \text{qd-dim } [2, \mathcal{B}] = 3 \quad [\text{qd-dim } \mathcal{B} = 0 = \text{qd-dim } \mathcal{B}_2]$$

Proof: By 3.1.2 it is sufficient to consider diagrams of the form $\mathcal{B}_k[A]$. It follows readily from [1, Lemma 3.3] that for $k > 4$, $\text{hd } \mathcal{B}_k[A] < 3$ for A in $\mathcal{B}$ or $\mathcal{B}_2$. We shall show for $k \neq 0$, that, if A is in $\mathcal{B}$, then $\text{hd } \mathcal{B}_k[A] = 3$, and if A is in $\mathcal{B}_2$, then $\text{hd } \mathcal{B}_k[A] = 4$.

Consider

$$(1) \quad 0 \longrightarrow \mathcal{B}^2 \longrightarrow \mathcal{B}^2 \xrightarrow{\mathcal{A}^2} \mathcal{B}^1 \xrightarrow{\mathcal{A}^1} \mathcal{B}^0 \xrightarrow{\mathcal{A}^0} \mathcal{B}_k[A] \longrightarrow 0$$

where $\mathcal{B}^2 = \mathcal{B}_k[A]$, $\mathcal{B}^1 = \bigoplus_{i=1}^3 \mathcal{B}_{k-1}[A]$, $\mathcal{B}^0 = \bigoplus_{i=1}^3 \mathcal{B}_{k-1}(\langle \mathcal{A}_{k-1}^1 \rangle)$, $\mathcal{B}^2$ is the kernel of $\mathcal{A}^1$, and the maps $\mathcal{A}^i$ ($i = 0, 1, 2$) are the induced maps. Clearly $\mathcal{A}^2$ is projection for $i = 0, 1, 2$, and it is straightforward to verify that (1) is exact.

By $\mathcal{A}\mathcal{B}_1 \oplus \mathcal{A}\mathcal{B}_2 \oplus \mathcal{A}\mathcal{B}_3$ we mean $A \oplus A \oplus A$, and by $\mathcal{A}\mathcal{B}_1 \oplus \mathcal{A}\mathcal{B}_2$ we mean $A \oplus 0 \oplus A$ considered as a subspace (subobject) of $A \oplus A \oplus A$ in the obvious way. Similarly, for simplicity, we shall denote an element of $\mathcal{B}_{k-1}^1 \mathcal{B}_1 \oplus \mathcal{B}_{k-1}^1 \mathcal{B}_2$ by $\mathcal{B}_1 \mathcal{B}_1 + \mathcal{B}_2 \mathcal{B}_2$, where $\mathcal{B}_1 \in \mathcal{B}_{k-1}^1$ and $\mathcal{B}_2 \in \mathcal{B}_{k-1}^1$, etc.

The exact sequence (1) may be displayed in a more detailed fashion as follows.

$H_k(A)$	$\mathbb{Z}^0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$
d	$\mathbb{Z}$	$\mathbb{Z}$	0
a_1	0	$\mathbb{Z}$	$\mathbb{Z}a_1$
a_2	0	$\mathbb{Z}$	$\mathbb{Z}a_2$
a_3	0	$\mathbb{Z}$	$\mathbb{Z}a_3$
b_1	0	$\mathbb{Z}$	$\mathbb{Z}a_1 \oplus \mathbb{Z}a_2$
b_2	0	$\mathbb{Z}$	$\mathbb{Z}a_2 \oplus \mathbb{Z}a_3$
b_3	0	$\mathbb{Z}$	$\mathbb{Z}a_1 \oplus \mathbb{Z}a_3$
b_4	0	$\mathbb{Z}$	$\mathbb{Z}a_1 \oplus \mathbb{Z}a_3$
b_5	0	$\mathbb{Z}$	$\mathbb{Z}a_1 \oplus \mathbb{Z}a_2$
b_6	0	$\mathbb{Z}$	$\mathbb{Z}a_2 \oplus \mathbb{Z}a_3$
c_1	0	$\mathbb{Z}$	$\mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \oplus \mathbb{Z}a_3$
c_2	0	$\mathbb{Z}$	$\mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \oplus \mathbb{Z}a_3$
c_3	0	$\mathbb{Z}$	$\mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \oplus \mathbb{Z}a_3$
c_4	0	$\mathbb{Z}$	$\mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \oplus \mathbb{Z}a_3$
e	0	$\mathbb{Z}$	$\mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \oplus \mathbb{Z}a_3$

$$\bigoplus_{i=1}^4 \mathbb{Z}a_i$$

We shall complete the proof by showing: if $\lambda \neq 0$ is in $\mathbb{R}_2$, then $\mathcal{K}^2$ is not projective, and hence $\dim \mathcal{K}_\lambda(\lambda) = 0$ since clearly $\dim \mathcal{K}^2 = 1$; and if $\lambda \neq 0$ is in $\mathbb{R}_1$, then $\mathcal{K}^2$ is projective, and hence $\dim \mathcal{K}_\lambda(\lambda) = 3$ since $\mathcal{K}^1$ is clearly not projective.

Show that $\mathcal{K}^2$ is projective if and only if $\{\mathcal{K}_{\alpha_j}^2 : j = 1, 2, 3, 4\}$ is an independent family. One may easily verify that if $\lambda \neq 0$ is in $\mathbb{R}_1$ or $\mathbb{R}_2$, then $\{\mathcal{K}_{\alpha_1}^2, \mathcal{K}_{\alpha_2}^2, \mathcal{K}_{\alpha_3}^2\}$ is an independent family. To illustrate this, we verify that

$$(8) \quad \mathcal{K}_{\alpha_1}^2 \cap (\mathcal{K}_{\alpha_2}^2 \oplus \mathcal{K}_{\alpha_3}^2) = \emptyset$$

If γ is in the left member of (8), then

$$\begin{aligned} \gamma &= \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 \\ &= (\beta_1 \alpha_2 + \alpha_2 \beta_1 + \beta_2 \beta_3) + (\alpha_1 \alpha_3 + \alpha_2 \alpha_3 + \alpha_3 \beta_3), \end{aligned}$$

where $\alpha_1 \in \mathcal{K}_{\alpha_1}^1$, $\alpha_2 \in \mathcal{K}_{\alpha_2}^1$, etc.; thus

$$0 = \alpha_1 = \beta_1 = \alpha_2 = \beta_2,$$

now since γ is in $\mathcal{K}_{\alpha_2}^2$, the kernel of α_2^2 ,

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

and similarly

$$\beta_1 = \beta_2 = \beta_3 = 0.$$

Hence

$$\alpha_2 = \alpha_3 = 0 = \beta_2 = \beta_3.$$

Thus

$$\alpha_3 = -\alpha_2 = -\beta_2 = \beta_3.$$

However, $x_3 \notin K_{\alpha_0}^1$, $x_3 \notin K_{\alpha_0}^2$, and $K_{\alpha_0}^1 \cap K_{\alpha_0}^2 = \emptyset$. Therefore $\tau = 0$. Note that the preceding argument is valid if λ is in $\mathbb{R}$ or $\mathbb{C}_{\geq 0}$.

Now we observe that

$$(3) \quad (x_{\alpha_i}^2 : i = 1, 2, 3, 4) \text{ is an independent family if and only if}$$

$$x_{\alpha_1}^2 \cap (x_{\alpha_2}^2 \oplus x_{\alpha_3}^2 \oplus x_{\alpha_4}^2) = \emptyset.$$

But (3) holds if and only if $x_1 = x_2 = x_3 = 0$ is implied by

(4) the existence of $x_1 \in K_{\alpha_1}^1$, $x_2 \in K_{\alpha_2}^1$, $x_3 \in K_{\alpha_3}^1$, $x_4 \in K_{\alpha_4}^1$, $x_1 \in K_{\alpha_2}^2$, etc., such that

$$x_1 x_1 + x_2 x_2 + x_3 x_3 = (x_1 x_1 + x_2 x_1 + x_3 x_1) +$$

$$(x_2 x_2 + x_2 x_1 + x_3 x_1) + (x_1 x_3 + x_2 x_3 + x_3 x_3)$$

and

$$x_1 + x_2 + x_3 = x_1 + x_2 + x_3$$

$$= x_1 + x_2 + x_3 + x_1 + x_2 + x_3 = 0.$$

Moreover, (4) is true if and only if

$$(5) \quad \text{There is } x_1 \in K_{\alpha_1}^1, x_2 \in K_{\alpha_2}^1, \text{ and } x_3 \in K_{\alpha_3}^1 \text{ such that}$$

$$x_1 + x_2 \in K_{\alpha_3}^1, x_1 + x_3 \in K_{\alpha_2}^1, \text{ and } x_2 + x_3 \in K_{\alpha_1}^1.$$

Also, (5) is true if and only if

$$(6) \quad \text{There is a } \lambda \in \mathbb{R} \text{ such that } x_1 = -\lambda \text{ and}$$

$$x_1 = \alpha x_1 = \alpha x_2$$

$$x_2 = \alpha x_2 = \alpha x_3$$

$$x_3 = \alpha x_1 = \alpha x_2$$

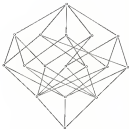
Now if $A \neq \emptyset$ is in $\mathcal{B}_2$, then there is a $\beta \in A$ such that (8) holds and $\alpha_1 \neq \alpha$, since (5) is equivalent to (8), we have that (3) does not hold.

On the other hand, if $A \neq \emptyset$ is in $\mathcal{B}$ and (4) holds, then (8) holds. But $\alpha \in A$ and $\alpha = -\alpha$ imply $\alpha = \emptyset$, which implies $\alpha_1 = \alpha_2 = \alpha_3 = \emptyset$. Thus (3) holds.

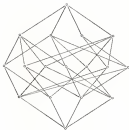
(5.1.4) 5.1.2 is not an isolated counterexample, although it seems to be the smallest. In fact, we have all but filled in the details of a proof that there are infinitely many counterexamples like 5.1.2. That is, for each set I , $\text{gl.dim } [I, \mathcal{B}] = 3$ and $\text{gl.dim } [I, \mathcal{B}_2] = 4$, and also, no proper subset of I has these properties.

We give two such partially ordered sets,

(5.2.1)



(5.8.2)



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BIOGRAPHICAL SKETCH

William Charles Sprues was born at Sebring, Florida, on October 27, 1943. He graduated from Charlotte-Mecklenburg High School at Sebring, Florida, in June of 1961, and entered the University of Florida in September of 1961. He received his Bachelor of Science degree from the University of Florida in December of 1964, and entered the Graduate School in January, 1965. In December of 1968, he received his Master of Arts degree from the University of Florida.

This dissertation was prepared under the direction of the advisors of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Senate of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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